Foreword

In this book one finds theorems on which arithmetic is based, proven using signs that collectively I call concept-script. The most important of these propositions, some with an accompanying translation appended, are listed at the end. As may be seen, the investigation does not yet include the negative, rational, irrational and complex numbers, nor addition, multiplication, etc. Moreover, propositions about the cardinal numbers are not yet present with the completeness initially planned. Missing, in particular, is the proposition that the cardinal number of objects falling under a concept is finite, if the cardinal number of objects falling under a superordinate concept is finite. External reasons have made me postpone both this and the treatment of other numbers, and mathematical operations, to a sequel whose publication will depend on the reception of this first volume. What I have offered here may suffice to give an idea of my method. It might be thought that the propositions concerning the cardinal number Endlos[1] could have been omitted. To be sure, they are not needed for the foundation of arithmetic in its traditional extent; but their derivation is often easier than those of the corresponding propositions concerning finite cardinal numbers and can serve as preparation for the latter. Propositions also occur which are not about cardinal numbers but which are needed in proofs. They treat, for example, of following in a series, of single-valuedness of relations, of composite and coupled relations, of mapping by means of relations, and such like. These propositions could perhaps be allocated to an extended theory of combinations.

The proofs are contained solely in the sections entitled "Construction", while those headed "Analysis" are meant to facilitate understanding by providing a preliminary and rough sketch of the proof. The proofs themselves contain no words but are carried out solely in my symbolism. They are presented as a series of formulae separated by

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[1] Cardinal number of a countably infinite set.

continuous or broken lines or other signs. Each of these formulae is a complete proposition displaying all the conditions on which its validity depends. This completeness, which does not tolerate any tacit addition of assumptions in thought, seems to me indispensable for the rigorous conduct of proof.

The progression from one proposition to the next proceeds by the rules which are listed in §48, and no transition is made that does not accord with these rules. How, and according to which rule, an inference is drawn is indicated by the sign standing between the formulae, while ——— • ——— marks the termination of a chain of inferences. For this purpose there have to be propositions which are not derived from others. Some of these are the basic laws listed in §47; others are definitions which are collected in a table at the end, together with a reference to their first occurrence. Time and again, the pursuit of this project will generate a need for definitions. Their governing principles are listed in §33. Definitions themselves are not creative, and in my view must not be; they merely introduce abbreviative notations (names), which could be dispensed with were it not for the insurmountable external difficulties that the resulting prolixity would cause.

The ideal of a rigorous scientific method for mathematics that I have striven to realise here, and which could be named after Euclid, can be characterised as follows. It cannot be required that everything be proven, as this is impossible; but it can be demanded that all propositions appealed to without proof are explicitly declared as such, so that it can be clearly recognised on what the whole structure rests. One must strive to reduce the number[a] of these fundamental laws as far as possible by proving everything that is provable. Furthermore, and in this I go beyond Euclid, I demand that all modes of inference and consequence which are used be listed in advance. Otherwise compliance with the first demand cannot be secured. This ideal I believe I have now essentially achieved. Only in a few points could one impose even more rigorous demands. In order to attain more flexibility and to avoid excessive length, I have allowed myself tacit use of permutation of subcomponents (conditions) and fusion of equal subcomponents, and have not reduced the modes of inference and consequence to a minimum. Anyone acquainted with my little book *Begriffsschrift* will gather from it how here too one could satisfy the strictest demands, but also that this would result in a considerable increase in extent.

Furthermore, I believe that the criticisms that can justifiably be made of this book

will pertain not to rigour but rather only to the choice of the course of proof and of the intermediate steps. Often several ways of conducting a proof are available; I have not tried to pursue them all and it is possible, indeed likely, that I have not always chosen the shortest. Let whoever has complaints on this score try to do better. Other matters will be disputable. Some might have preferred to increase the circle of permissible modes of inference and consequence, in order to achieve greater flexibility and brevity. However, one has to draw a line somewhere if one approves of my stated ideal at all; and wherever one does so, people could always say: it would have been better to allow even more modes of inference.

The gaplessness of the chains of inferences contrives to bring to light each axiom, each presupposition, hypothesis, or whatever one may want to call that on which a proof rests; and thus we gain a basis for an assessment of the epistemological nature of the proven law. Although it has already been announced many times that arithmetic is merely logic further developed, still this remains disputable as long as there occur transitions in the proofs which do not conform to acknowledged logical laws but rather seem to rest on intuitive knowledge.[b] Only when these transitions are analysed into simple logical steps can one be convinced that nothing but logic forms the basis. I have listed everything that can facilitate an assessment whether the chains of inferences are properly connected and the buttresses are solid. If anyone should believe that there is some fault, then he must be able to state precisely where, in his view, the error lies: with the basic laws, with the definitions, or with the rules or a specific application of them. If everything is considered to be in good order, one thereby knows precisely the grounds on which each individual theorem rests. As far as I can see, a dispute can arise only concerning my basic law of value-ranges (V), which perhaps has not yet been explicitly formulated by logicians although one thinks in accordance with it if, e.g., one speaks of extensions of concepts. I take it to be purely logical. At any rate, the place is hereby marked where there has to be a decision.

My purpose demands some divergences from what is common in mathematics. Rigour of proof requires, as an inescapable consequence, an increase in length. Whoever fails to keep an eye on this will indeed be surprised how cumbersome our proofs often are of propositions into which he would suppose he had an immediate insight, through a single act of cognition.[c] This will be especially striking if one compares Mr Dedekind's essay, *Was sind und was sollen die Zahlen?*, the most thorough study I have seen in recent times concerning the foundations of arithmetic. It pursues, in

much less space, the laws of arithmetic to a much higher level than here. This concision is achieved, of course, only because much is not in fact proven at all. Often, Mr Dedekind merely states that a proof follows from such and such propositions; he uses dots, as in "$\mathfrak{M}(A, B, C \ldots)$"; nowhere in his essay do we find a list of the logical or other laws he takes as basic; and even if it were there, one would have no chance to verify whether in fact no other laws were used, since, for this, the proofs would have to be not merely indicated but carried out gaplessly. Mr Dedekind too is of the opinion that the theory of numbers is a part of logic; but his essay barely contributes to the confirmation of this opinion since his use of the expressions "system", "a thing belongs to a thing" are neither customary in logic nor reducible to something acknowledged as logical. I do not say this as a complaint; his procedure may have been the most appropriate for his purpose; I say this only to cast a brighter light upon my own intentions by contrast. The length of a proof should not be measured by the ell. It is easy to make a proof appear short on paper, by missing out many intermediate steps in the chain of inferences or by merely gesturing at them. Mostly, no doubt, one contents oneself with the obvious correctness of each step in a proof; and permissibly so, if the aim is merely to persuade of the truth of the proposition to be proven. However, if the aim is to convey insight into the nature of this obviousness, this procedure does not suffice; rather, one must write out all intermediate steps, so that the full light of awareness may fall upon them. Usually, mathematicians are merely concerned with the content of a proposition and that it be proven. Here the novelty is not the content of the proposition, but how its proof is conducted, on what foundations it rests. That this essentially different perspective also requires another kind of treatment must not put us off. When one of our propositions is proven in the usual manner, then a proposition that appears to be unnecessary for the proof will easily be overlooked. In a thorough examination of my proof given here, I believe, one will indeed realise its indispensability, unless an entirely different path is taken. Here and there one will perhaps also encounter conditions in our propositions that strike one as redundant at first, but which will prove to be necessary after all, or at least eliminable only by using a proposition to be proven for this specific purpose.

I here carry out a project that I already had in mind at the time of my *Begriffsschrift* of the year 1879 and which I announced in my *Grundlagen der Arithmetik* of the year 1884.[1] By this act I aim to confirm the conception of cardinal number

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[1] Compare the introduction and §§90 and 91 in my *Grundlagen der Arithmetik*, Breslau, Verlag von Wilhelm Koebner, 1884.

which I set forth in the latter book. The basis for my results is articulated there in §46, namely that a statement of number contains a predication about a concept; and the exposition here rests upon it. If someone takes a different view, he should try to develop a sound and usable symbolic exposition on that basis; he will find that it will not work. No doubt in language the point is not so transparent; but if one pays close attention, one finds that even here there is mention of a concept, rather than of a group, an aggregate or suchlike, whenever a statement of number is made; and even if exceptions sometimes occur, the group or the aggregate is always determined by a concept, i.e., by the properties an object must have in order to belong to the group, while what unites the members to each other, has absolutely no bearing on the cardinal number.

The reason why the implementation appears so late after the announcement is owing in part to internal changes within the concept-script which forced me to jettison a nearly completed handwritten work. This progress might be mentioned here briefly. The primitive signs used in my *Begriffsschrift* occur again here with one exception. Instead of the three parallel lines, I have chosen the usual equality-sign, for I have convinced myself that in arithmetic it possesses just that reference that I too want to designate. Thus, I use the word "equal" with the same reference as "coinciding with" or "identical with", and this is also how the equality-sign is actually used in arithmetic. The objection to this which might be raised would rest on insufficiently distinguishing between sign and what is designated. No doubt, in the equation '$2^2 = 2 + 2$' the sign on the left is different from the one on the right; but both designate or refer to the same number.[1] To the original primitive signs two have now been added: the smooth breathing, designating the value-range of a function, and a sign to play the role of the definite article in language. The introduction of value-ranges of functions is an essential step forward, thanks to which we achieve far greater flexibility. What previously had been derived signs can now be replaced by other, and indeed simpler, ones, although the definitions of single-valuedness of a relation, of following in a series, of mapping are essentially the same as those given partly in my *Begriffsschrift*, partly in my *Grundlagen der Arithmetik*. Value-ranges, however, have a much more

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[1] To be sure, I also say: the sense of the sign on the right is different from the one on the left; but the reference is the same. Compare my essay "Über Sinn und Bedeutung" in the *Zeitschrift f. Philos. u. philos. Kritik*, vol. 100, p. 25.

fundamental importance; for I define cardinal numbers themselves as extensions of concepts, and extensions of concepts are value-ranges, according to my specification. So without the latter one would never be able to get by. The old primitive signs that re-occur outwardly unaltered, and whose algorithm has hardly changed, have however been provided with different explanations. What was formerly the content-stroke reappears as the horizontal. These are consequences of a deep-reaching development in my logical views. Previously I distinguished two components in that whose external form is a declarative sentence: 1) acknowledgement of truth, 2) the content, which is acknowledged as true. The content I called judgeable content. This now splits for me into what I call thought and what I call truth-value. This is a consequence of the distinction between the sense and the reference of a sign. In this instance, the thought is the sense of a proposition and the truth-value is its reference. In addition, there is the acknowledgment that the truth-value is the True. For I distinguish two truth-values: the True and the False. I have justified this in more detail in my above mentioned essay *Über Sinn und Bedeutung*. Here, it might merely be mentioned that only in this way can indirect speech be accounted for correctly. For in indirect speech, the thought, which is normally the sense of the proposition, becomes its reference. Only a thorough engagement with the present work can teach how much simpler and more precise everything is made by the introduction of the truth-values. These advantages alone already weigh heavily in favour of my conception, which at first sight might admittedly seem strange. Moreover, the nature of functions, in contrast to objects, is characterised more precisely than in my *Begriffsschrift*. Further, from this the distinction between functions of first and second level results. As elaborated in my lecture *Function und Begriff*[1] concepts and relations are functions as I extend the reference of the term, and so we also must distinguish concepts of first and second level and relations of equal and unequal level.

As one can see, the years since the publication of my *Begriffsschrift* and *Grundlagen* have not passed in vain: they have seen the work mature. But the very thing which I regard as essential progress serves, as I cannot conceal from myself, as a major obstruction to the dissemination and influence of this book. Moreover, what I regard as not the least of its virtues, strict gaplessness of the chains of inferences, will earn it, I fear, scant appreciation. I have departed further from traditional conceptions

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[1] Jena, Verlag von Hermann Pohle.

and thereby impressed on my views a paradoxical character. An expression, cropping up here and there as one leafs through the pages, will all too easily seem strange and provoke negative prejudice. I can myself gauge somewhat the resistance which my innovations will encounter, as I too had first to overcome something similar in order to make them. To be sure, I have arrived at them not arbitrarily and out of a craze for novelty, but was forced by the very subject matter itself.

With this, I arrive at a second reason for the delay: the despondency that at times overcame me as a result of the cool reception, or rather, the lack of reception, by mathematicians[1] of the writings mentioned above, and the unfavourable scientific currents against which my book will have to struggle. The first impression alone can only be off-putting: strange signs, pages of nothing but alien formulae. Thus sometimes I concerned myself with other subjects. Yet as time passed, I simply could not contain these results of my thinking, which seemed to me valuable, locked up in my desk; and work expended always called for further work if it was not to be in vain. Thus the subject matter kept me captive. In such a case, when the value of a book cannot be appreciated on a swift reading, the reviewer should step in to assist. But in general the remuneration will be too poor. The critic can never hope to be compensated in money for the effort that a thoroughgoing study of this book will demand. All that is left for me is to hope that someone may from the outset have sufficient confidence in the work to anticipate that his inner reward will be repayment enough, and will then publicise the results of a thorough examination. It is not that only a complimentary review could satisfy me; quite the contrary! I would always prefer a critical assault based on a thorough study to praise that indulges in generalities without engaging the heart of the matter. Now I would like to offer some hints to assist the work of a reader approaching the book with these intentions.

In order to gain an initial rough idea of how I express thoughts with my signs, it will be helpful to look at some of the easier cases in the table of the more important theorems, to which a translation is appended. It will then be possible to surmise what is intended in further, similar examples which are not followed by a translation. Next, one should begin with the introduction and start to tackle the exposition of the concept-script. However, I advise first to make merely a summary overview of it

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[1] One searches in vain for my *Grundlagen der Arithmetik* in the *Jahrbuch über die Fortschritte der Mathematik*. Researchers in the same area, Mr Dedekind, Mr Otto Stolz, Mr von Helmholtz seem not to be acquainted with my works. Kronecker does not mention them in his essay on the concept of number either.

and not to dwell on particular concerns. In order to meet all objections, some issues have had to be taken up which are not required for understanding concept-script propositions. I include in this the second half of §8 which starts on p. 12 with "If we now give the following explanation", and also the second half of §9, which starts on p. 15 with the words "If I say in general", together with the whole of §10. These passages should be omitted on a first reading. The same applies to §26 and §§28–32.

By contrast, I wish to lay stress on the first half of §8, as well as §§12 and 13, as particularly important for understanding. A more detailed reading should start with §34 and continue to the end. Occasionally, one will have to revisit §§ merely fleetingly read. The index at the end and the table of contents will facilitate this. The derivations in §§49–52 can be used as preparation for an understanding of the proofs themselves. Here, all modes of inference and nearly all of the applications of our basic laws already occur. When one has reached the end, one should reread the entire exposition of the concept-script with this as background, keeping in mind that those stipulations that will not be used later, and therefore appear unnecessary, serve to implement the principle that all correctly formed signs ought to refer to something—a principle that is essential for full rigour. In this way, I believe, the mistrust that my innovations may initially provoke will gradually disappear. The reader will recognise that my principles will in no case lead to consequences other than ones he must acknowledge as correct himself. Perhaps he will then admit that he had overestimated the labour, that, in fact, my gapless approach facilitates understanding, once the barrier presented by the novelty of the signs is overcome. May I be fortunate enough to find such a reader or reviewer! For a review based on a superficial reading might easily do more harm than good.

Otherwise, of course, the prospects for my book are dim. In any case, we must give up on those mathematicians who, encountering logical expressions like "concept", "relation", "judgement", think: *metaphysica sunt, non leguntur!*'d and also on those philosophers who, sighting a formula, cry out: *mathematica sunt, non leguntur!* and the exceptions will be very few. Perhaps the number of mathematicians who care about the foundation of their science is not large in any case, and even these often seem to be in a great hurry until they leave the fundamentals behind them. Moreover, I hardly dare hope that many of them will be convinced by my reasons for the painstaking rigour, and the lengthiness connected with it. Custom exerts great

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power over the mind. If I compare arithmetic with a tree that high up unfolds in a multiplicity of methods and theorems, while the root stretches into the depths, then it seems to me that the growth of the root, at least in Germany, is weak. Even in the *Algebra der Logik* of Mr E. Schröder, a work one would want to count as pursuing this direction, upper growth soon dominates before any greater depth is attained, causing an upward bent and a ramification into methods and theorems.

Of further disadvantage for my book is a widespread tendency to accept only what can be sensed as being. What cannot be perceived with the senses one tries to disown, or at least to ignore. Now the objects of arithmetic, the numbers, are imperceptible; how to come to terms with this? Very simple! Declare the number-signs to be the numbers. In the signs, one then has something visible; and this, of course, is the main thing. To be sure, the signs have properties completely different from the numbers; but so what? Just credit them with the desired properties by so-called definitions. To be sure, it is a puzzle how there can be a definition where there is no question of a connection between sign and what is designated. One kneads together sign and what is designated as indistinguishably as possible; depending on what is required, one can assert existence by appeal to their tangibility[1] or bring the true properties of the numbers to the foreground. On occasion, it seems that the number-signs are regarded like chess pieces, and the so-called definitions like rules of the game. In that case the sign designates nothing, but is rather the thing itself. One small detail is overlooked in all this, of course; namely that a thought is expressed by means of '$3^2 + 4^2 = 5^2$', whereas a configuration of chess pieces says nothing. When one is content with such superficialities, there is surely no basis for a deeper understanding.

Here it is crucial to get clear about what definition is and what it can achieve. Often one seems to credit it with a creative power, although in truth nothing takes place except to make something prominent by demarcation and designate it with a name. Just as the geographer does not create a sea when he draws borderlines and says: the part of the water surface bordered by these lines I will call Yellow Sea, so too the mathematician cannot properly create anything by his definitions. Moreover, a property which a thing just does not have cannot be magically attached to it by mere definition, except for the property of now being called by the name that one has given to it. That, however, an egg-shaped figure, produced with ink on paper, may

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[1] Compare E. Heine, *Die Elemente der Functionslehre*, in Crelle's *Journal*, vol. 74, p. 173: "Concerning definitions, I take the purely formal standpoint in calling certain tangible signs numbers, so that the existence of these numbers is thus not in question."

be endowed by definition with the property of resulting in One if added to One, I can only regard as scientific superstition. A lazy student could just as well be turned into a diligent one by means of definition alone. Unclarity develops easily here for want of the distinction between concept and object. If one says: "A square is a rectangle in which adjacent sides are equal", then one defines the concept *square* by stating what properties something must have in order to fall under it. I call these properties characteristic marks of the concept. Yet, note that these characteristic marks of the concept are not its properties. The concept *square* is not a rectangle, it is only the objects that fall under this concept that are rectangles, just as the concept *black cloth* is neither black nor a cloth. Whether there are such objects is not immediately known on the basis of the definition. One wants to define the number Zero, for example, by saying: it is something which when added to One, results in One. Thus a concept is defined by stating what property an object must have in order to fall under it. This property, however, is not a property of the defined concept. Yet, as it seems, it is often imagined that something which added to One results in One is created by definition. What a great illusion! The defined concept does not possess this property, nor does the definition guarantee that the concept is instantiated. This first requires an investigation. Only when one has shown that there is one and only one object with the requisite property is one in a position to give this object the proper name "Zero". To create Zero is hence impossible. I have repeatedly spelt these things out but, seemingly, without success.[1]

A proper appreciation of the distinction I draw, between a characteristic mark of a concept and a property of an object, can scarcely be hoped for from the prevailing logic either,[2] for that seems to be contaminated with psychology through and through. If instead of the things themselves, one considers only their subjective images, their ideas, then naturally all finer-grained, objective distinctions are lost and others appear in their place that are logically completely worthless. Thus I come to speak about the obstacle to the influence of my book on the logicians. It is the ruinous incursion of psychology into logic. Decisive for the treatment of this science is how the logical laws are conceived, and this in turn connects with how one understands

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[1] Mathematicians who prefer not to enter into the mazes of philosophy are requested to stop reading the foreword here.
[2] In the logic of Mr B. Erdmann I find no trace of this important distinction.

the word "true". It is commonly granted that the logical laws are guidelines which thought should follow to arrive at the truth; but it is too easily forgotten. The ambiguity of the word "law" here is fatal. In one sense it says what is, in the other it prescribes what ought to be. Only in the latter sense can the logical laws be called laws of thought, in so far as they legislate how one ought to think. Every law stating what is the case can be conceived as prescriptive, one should think in accordance with it, and in that sense it is accordingly a law of thought. This holds for geometrical and physical laws no less than for the logical. The latter better deserve the title "laws of thought" only if thereby it is supposed to be said that they are the most general laws, prescribing how to think wherever there is thinking at all. But the phrase "laws of thought" seduces one to form the opinion that these laws govern thinking in the same way that the laws of nature govern events in the external world. In that case they can be nothing other than psychological laws; for thinking is a mental process. And if logic had to do with psychological laws, it would be a part of psychology. And thus it is in fact conceived. These laws of thought may then be conceived as guidelines merely in the manner of stating a mean, similar to the way one can say how healthy digestion proceeds in humans, or how grammatically correct speech goes, or how one dresses fashionably. Then one can merely say: humans' taking to be true conforms in average to these laws, both at present and wherever human beings are found; so, if one wants to stay in harmony with the mean, one had better follow suit. However, what's fashionable today will be out of fashion sometime, and is at present not fashionable amongst the Chinese; so, likewise, one can present psychological laws of thought as setting a standard only with restrictions. Indeed so, if logic deals with being taken to be true and not, rather, with being true! And that is what the psychological logicians conflate. Thus in the first volume of his *Logik*,[1] pp. 272 to 275, Mr B. Erdmann equates truth with general validity, grounding the latter on general certainty regarding the object judged, and this in turn on general consensus amongst those judging. And so, in the end, truth is reduced to being taken to be true by individuals. In opposition to this, I can only say: being true is different from being taken to be true, be it by one, be it by many, be it by all, and is in no way reducible to it. It is no contradiction

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[1] Halle a. S., Max Niemeyer, 1892.

that something is true that is universally held to be false. By logical laws I do not understand psychological laws of taking to be true, but laws of being true. If it is true that I write this in my room on 13th July, 1893, while the wind is howling outside, then it remains true even if all humans should later hold it to be false. If being true is thus independent of anyone's acknowledgement, then the laws of being true are not psychological laws either but boundary stones which are anchored in an eternal ground, which our thinking may wash over but yet cannot displace. And because of this they set the standards for our thinking if it wants to attain the truth. Their relation to thinking is not like that of the grammatical laws to language, as if they were to give expression to the nature of our human thinking and vary with it. The conception of the logical laws according to Mr Erdmann is, of course, entirely different. He doubts their unconditional, eternal validity and wants to restrict them to our thinking as it is now (pp. 375ff). But "our thinking" can surely only mean the thinking of humanity up until now. Accordingly, the possibility remains open that human or other beings might be discovered who could execute judgements contradicting our logical laws. What if this were to happen? Mr Erdmann would say: so we see that those principles are not valid everywhere. Certainly! if they are to be psychological laws, they ought to be formulated in a way that makes explicit the genus of beings whose thinking is empirically governed by them. I would say: there are therefore beings who do not recognise certain truths immediately in the manner we do but are reliant, perhaps, on the more protracted way of induction. What, however, if beings were even found whose laws of thought directly contradicted ours, so that their application often led to opposite results? The psychological logician could only accept this and say: for them, those laws hold, for us these. I would say: here we have a hitherto unknown kind of madness. He who thinks of logical laws as prescriptive of what ought to be thought, or as laws of what is true, rather than as natural laws concerning humans' taking to be true, will ask: Who is right? Whose laws of taking to be true are in accord with the laws of being true? The psychological logician cannot admit this question; for by so doing he would acknowledge laws of being true that were not psychological. Can the sense of the word "true" be subjected to a more damaging corruption than by the attempt to incorporate a relation to the judging subject! Surely no-one will here object that the proposition "I am hungry" could be true for one but false for another? The proposition, no doubt, but not the thought; for the word "I" in the mouth of the other refers to a different person,

and the proposition, accordingly, expresses a different thought when it is uttered by him. All determinations of place, time, and so on, belong to the thought whose truth is at issue; being true itself is place- and timeless. How, then, is the principle of identity to be read? Is it like this: "It is impossible for humans in the year 1893 to acknowledge an object as being different from itself"? Or like this: "Every object is identical to itself"? The former law is about humans and contains a determination of time; in the latter, there is mention neither of humans nor of time. The latter is a law of being true; the former one of human taking to be true. Their content is entirely different, and they are independent of each other so that neither can be inferred from the other. This is why it is very confusing to designate both by the same name of the basic law of identity. Such confusions of fundamentally different things are to blame for the appalling unclarity which we find in the psychological logicians.

As to the question, why and with what right we acknowledge a logical law to be true, logic can respond only by reducing it to other logical laws. Where this is not possible, it can give no answer. Stepping outside logic, one can say: our nature and external circumstances force us to judge, and when we judge we cannot discard this law—of identity, for example—but have to acknowledge it if we do not want to lead our thinking into confusion and in the end abandon judgement altogether. I neither want to dispute nor to endorse this opinion, but merely note that what we have here is not a logical conclusion. What is offered here is not a ground of being true but of our taking to be true. And further: this impossibility, to which we are subject, of rejecting the law does not prevent us from supposing beings who do so; but it does prevent us from supposing that such beings do so rightly; and it prevents us, moreover, from doubting whether it is we or they who are right. At least this is true of myself. If others dare in the same breath to both acknowledge a law and doubt it, then that seems to me to be an attempt to jump out of one's own skin against which I can only urgently warn. Whoever has once acknowledged a law of being true has thereby also acknowledged a law that prescribes what ought to be judged, wherever, whenever and by whomsoever the judgement may be made.

Surveying the whole matter, it seems to me that different conceptions of truth lie at the source of the dispute. For me, truth is something objective, independent of the judging subject, for psychological logicians, it is not. What Mr B. Erdmann calls

"objective certainty" is only a general acknowledgement by those who judge and cannot, accordingly, be independent of them but is liable to change with their mental nature.

We can capture this more generally still: I acknowledge a realm of the objective, non-actual, while the psychological logicians take the non-actual to be subjective without further ado. Yet it is utterly incomprehensible why something that has being independently of the judging subject has to be actual, i.e., has to be capable of acting, directly or indirectly, upon the senses. No such connection between the concepts is to be found. One can even give examples to show the opposite. The number One, e.g., is not easily regarded as actual, unless one is a follower of J. S. Mill. On the other hand, it is impossible to credit each human with his own number One; for in that case we should first have to investigate to what extent the properties of these Ones agreed. And if someone said, "One times One is One", and another, "One times One is Two", then we could only register the difference and say: your One has that property, mine this. There could be no talk of a dispute about who is right or of an attempt to instruct; for there is no common object. Obviously this runs entirely contrary to the sense of the word "One" and the sense of the proposition "One times One is One". Since One, as the same for everybody, confronts everyone in the same way, it can no more be investigated by means of psychological observation than the Moon. Should there after all be ideas of the number One in individual minds, then these are still to be distinguished from the number One, just as ideas of the Moon are to be distinguished from the Moon itself. Since the psychological logicians fail to appreciate the possibility of the objective non-actual, they take concepts to be ideas and thereby assign them to psychology. But the true state of affairs asserts itself too forcefully for this to be accomplished easily. And hence a vacillation afflicts the use of the word "idea", so that sometimes it seems to refer to something which belongs to the mental life of the individual and which, in accordance with the psychological laws, amalgamates with other ideas, associates with them; while at other times, to something that confronts everyone in the same way, so that no bearer of ideas[f] is either mentioned or even presupposed. These two uses are incompatible; for the former, associations, amalgamations merely occur within the individual bearer of ideas and merely occur at something that is as private to the bearer of ideas as his joy or pain. It must never be forgotten that the ideas of different people, however similar they may be, which, by the way, we cannot ascertain precisely, nevertheless do not coincide but are to be distinguished. Everyone has his own ideas which cannot also belong to another. Here, of course, I understand "idea" in the psychological

sense. The vacillating use of the word causes unclarity and helps the psychological logicians conceal their weakness. When will this finally be put to an end! This way everything will eventually be dragged down into the realm of psychology; the boundary between the objective and the subjective is eroded further and further, and even actual objects are treated psychologically as ideas. For what is *actual* other than a predicate? And what are logical predicates other than ideas? Everything leads thus into idealism and therefore, as an unavoidable consequence, into solipsism. If everyone designated something different by the name "Moon", namely one of his ideas, much like he voices his pain with the exclamation "ouch!", then of course a psychological viewpoint would be justified; but a dispute concerning the properties of the Moon would be pointless: one could perfectly well assert of his moon the opposite of what another says of his with the same right. If we could apprehend nothing but what is internal to ourselves, then a conflict of opinion, a mutual understanding would be impossible since a common ground would be lacking, and such a common ground cannot be an idea in the sense of psychology. There would be no logic appointed to be arbiter in a conflict of opinions.

But lest I give the impression that I am tilting at windmills, let me illustrate this inescapable sinking into idealism with reference to a particular book. For this, I choose Mr B. Erdmann's above mentioned *Logik* as one of the most recent works of the psychological trend, one which might not be denied all significance. First, let us observe the following proposition (1, p. 85):

"Thus psychology teaches with certainty that the objects of memory and imagination, just as those of deranged hallucinatory and illusionary ideation,[8] are of an ideal nature. . . . Ideal, moreover, is the whole range of properly mathematical ideas, from the number-series down to the objects of mechanics."

What a motley! So, the number Ten should stand on the same level as hallucinations! Here obviously the objective non-actual is being conflated with the subjective. Some objective things are actual, others not. *Actual* is only one of many predicates and is of no more concern to logic than, for instance, the predicate *algebraic* as applied to a curve. Naturally, this conflation ensnares Mr Erdmann in metaphysics, however much he strives to distance himself from it. I take it to be a sure sign of error should logic have to rely on metaphysics and psychology, sciences which themselves require logical principles. Where in that case is the real basic ground on which everything rests? Or is the situation like that of Münchhausen who pulled himself out of the bog by

his own hair? I strongly doubt that this is possible and surmise that Mr Erdmann remains enmired in the psychologico-metaphysical bog.

There is no real objectivity for Mr Erdmann; for everything is idea. Let us convince ourselves of this on the basis of his own statements. We read on p. 187 of the first volume:

"As a relation between what is ideated,[h] a judgement presupposes at least two relata between which the relation holds. As a *predication* about what is ideated, it demands that one of these relata be determined as the object of which is predicated, the subject, ... the other as the object that is predicated, the predicate ..."[i]

To begin with, we see here that both the subject of the predication and the predicate are designated as object or what is ideated. Here "what is ideated" could have been written instead of "object", for we read (1, p. 81): "For objects are what is ideated." And also conversely, everything ideated is meant to be object. On p. 38 one finds:

"According to its origin, the ideated divides into objects of sense perception and self-consciousness on the one hand, and into primitive and derived on the other."

But what has its source in sense perception or self-consciousness is of course mental in nature. The objects, what is ideated, and hence also subject and predicate, are thereby assigned to psychology. This is confirmed by the following passage (1, pp. 147 and 148):

"It is the ideated or the idea in general. For both are one and the same: the ideated is the idea, the idea what is ideated."

The word "idea" is indeed usually taken in a psychological sense; that this is also Mr Erdmann's use can be seen from the passages:

"Consciousness therefore is the genus of feeling, ideation, wanting" (p. 35)

and

"Ideation is composed of the ideas ... and the passages of ideas" (p. 36).[j]

After this we should not be surprised that an object comes into being in a psychological manner:

"Insofar as a perception-mass ... presents the same as earlier stimuli and the excitations triggered by them, it *reproduces* the memory traces that originated from this same of the earlier stimuli and *amalgamates* with them into an object of the apperceived idea" (1, p. 42).[k]

On p. 43 it is then shown by way of an example how a steel engraving of Raphael's Sistine Madonna comes into being in a purely psychological way, without steel press,

ink and paper. After all this, no doubt can remain that the object about which a predication is made, the subject, is in Mr Erdmann's opinion taken to be an idea in the psychological sense of the word, as is the predicate, the object that is predicated. If this were right, then it could not be truthfully predicated of any subject that it was green, since there are no green ideas. Moreover, I could not predicate of any subject that it was independent of its being ideated or of myself, the bearer of ideas, any more than my decisions can be independent of my wanting and of myself, the wanting subject; rather they would be destroyed with me, if I were destroyed. So there is no real objectivity for Mr Erdmann, which follows also from his taking the ideated or ideas in general, objects in the most general sense of the word, as highest genus (γενικώτατον, *genus summum*) (p. 147). He is thus an idealist. If the idealists were consistent, they would regard the proposition "Charlemagne conquered the Saxons" neither as true nor as false but as fiction, just as we are accustomed to understand, for example, the proposition "Nessus carried Deïanira across the river Euenus"; for the proposition "Nessus did not carry Deïanira across the river Euenus" could likewise only be true if the name "Nessus" had a bearer. It would probably not be straightforward to drive the idealists out of this point of view. But one does not have to tolerate that they corrupt the sense of the proposition in this way, as if I wanted to predicate something of my idea when I speak of Charlemagne; what I want is to designate a man who is independent of myself and my ideation and to predicate something of him. One can grant the idealists that the achievement of this intention is not entirely certain, that without wanting to, I perhaps lapse from truth into fiction. But this has no bearing on the sense. With the proposition "This blade of grass is green", I predicate nothing of any idea of mine; I am not designating any of my ideas by means of the words "this blade of grass"; and were I doing so, the proposition would be false. At this point a second falsification intrudes, namely, that my idea of the green is being predicated of my idea of this blade of grass. I repeat: there is in no way any mention of my ideas in this proposition; an entirely different sense is being smuggled in here. Incidentally, I do not understand at all how an idea can be predicated of something. It would equally be a falsification if one were to say that in the proposition "The Moon is independent of me and my ideation", my idea of independence of myself and my ideation is predicated of my idea of the Moon. This would be to surrender objectivity in the proper sense of the word, and to put something entirely different in its place. No doubt it is possible that, in making

a judgement, such a play of ideas should occur; but that is not the sense of the proposition. It may also be observed that for one and the same proposition, and one and the same sense of the proposition, the play of ideas can be entirely different. Yet it is this logically irrelevant side-show which our logicians take as the proper object of their research.

How understandable it is that the nature of the subject matter recoils against sinking into idealism, and that Mr Erdmann does not want to admit that, for him, there is no real objectivity; but equally understandable is the futility of his endeavor. For if all subjects and predicates are ideas, and if all thinking is nothing but production, connection, change of ideas, then it is impossible to see how anything objective can ever be achieved. An indication of this futile resistance is the very use of the words "what is ideated" and "object" which at first apparently designate something objective, rather than an idea, but only apparently; for it becomes manifest that they refer to the same. To what purpose, then, this superfluity of expressions? This is not hard to guess. One may notice in addition that there is mention of the object of an idea, although the object is taken to be itself an idea. That would then be an idea of an idea. What relation between ideas might be designated by this? Unclear as this is, it is intelligible enough how, in the clash between the nature of the subject matter and idealism, such maelstroms can arise. Everywhere, we find the object of which I form an idea confused with this idea itself, only for their differences to come into prominence later. This conflict is manifest in the following proposition:

"For an idea whose object is general is thus, as such, as an event of consciousness, no more general than an idea itself is real because its object is posited as real, or than an object that we experience as sweet ... is presented by ideas which themselves are sweet" (I, p. 86).[1]

Here, the true state of affairs asserts itself with force. I could almost agree; but note that according to Erdmann's principles the object of an idea and the object which is presented by ideas are themselves ideas; and so we can see that all struggle is futile. Further, I ask to keep in mind the words "as such" that are similarly used on p. 83 in the following passage:

"When actuality is predicated of an object, the real subject of a judgement is not the object or the ideated as such but is rather *the transcendent*, which is presupposed as the ground of being of the ideated, through which the ideated presents itself. Here, the transcendent should not be regarded

as the unknowable ... rather its transcendence is only to consist in its independence from being ideated."[m]

Again, a vain attempt to haul oneself out of the bog! If we take these words seriously, then it is claimed that in this case the subject is not an idea. Yet if that is possible, then it cannot be seen why with other predicates, which express specific kinds of efficacy or actuality, the real subject must surely be an idea, e.g., as in the judgement "the earth is magnetic". So we would then arrive at the view that the real subject will be an idea in only a few judgements. However, once it is granted that it is not essential for either the subject or the predicate to be an idea, the rug is pulled out from under the whole psychological logic. All psychological considerations, which now swell our logic texts, thus prove to be pointless.

In fact, however, we probably should not take Mr Erdmann's notion of transcendence too seriously. I merely have to remind him of one of his statements (I, p. 148):

"Also subordinate to the highest genus is the *metaphysical* limit of our ideation, the transcendent",

and he is sunk; for the highest genus (γενικώτατον, *genus summum*) is, according to him, just the ideated, or the idea in general. Or might the word "transcendent" be used above in a different sense from here? In any case, one would suppose, the transcendent should be subordinate to the highest genus.

Let us dwell a moment longer on the expression "as such". I take the case where someone wants me to think that all objects are nothing but images on the retina of my eyes. Very well! I make no comment yet. But now he maintains that the tower is bigger than the window through which I take myself to be seeing it. To this, I would then say: either not both the tower and the window are retinal images in my eye, in which case the tower may be bigger than the window; or the tower and the window are, as you say, images on my retina, in which case the tower is not bigger but, rather, smaller than the window. At this point, he tries to relieve his embarrassment by resort to "as such", and says: the retinal image of the tower as such is, admittedly, not bigger than that of the window. Here I almost want to jump out of my skin and shout at him: well, in that case the retinal image of the tower is not at all bigger than that of the window; and if the tower were the retinal image of the tower and the window were the retinal image of the window, then the tower simply would not be bigger than the window, and if your logic teaches you otherwise, then it is good for nothing. This "as such" is an excellent invention of unclear writers who want to say neither yes nor no. However I do not tolerate such wavering between

the two, but rather ask: if actuality is predicated of an object, is the real subject of the judgement the idea, yes or no? If not, then it arguably is the transcendent, which is presupposed as the ground of being of such an idea. But the transcendent is itself what is ideated or an idea. Thus we are driven to assume that the subject of the judgement is not the ideated transcendent, but rather the transcendent which is presupposed as the ground of being of this ideated transcendent. So we would have to go on forever; and no matter how far we were to go, we could never get past the subjective. Incidentally, the same game could also be initiated with the predicate, and not only with the predicate *actual* but just as well with, for example, *sweet*. We should then first say: if one predicates actuality or sweetness, but rather the transcendent which is presupposed as ground of the ideated. Yet we would not be able to come to rest with this, but would always be driven further. What can we learn from this? That psychological logic is on the wrong track when it conceives of the subject and predicate of judgements as ideas in the psychological sense, that psychological considerations are no more appropriate in logic than in astronomy and geology. If we ever want to get past the subjective, then we have to think of cognition[n] as an activity that does not create what is cognised, but grasps what is already there. The image of grasping is well suited to elucidate the issue. When I grasp a pencil, many things take place in my body: stimulation of the nerves, changes in the tension and the pressure of muscles, tendons and bones, changes in the circulation of the blood. The sum of these processes, however, is not the pencil, nor do they create it. The latter has being independently of these processes. It is essential to grasping that there is something which is grasped; the inner changes alone are not the grasping. Similarly, what we mentally apprehend has being independently of this activity, of the ideas and their changes that are part of or accompany the apprehension; it is neither the sum of these processes nor is it created as part of our mental life.

Let us see further how subtler differences in the subject matter are smudged over by the psychological logicians. The point was already made in the case of character-istic mark and property. This is connected with the distinction I have emphasised between object and concept, as well as that between concepts of first and second level. Naturally, these differences are indiscernible by psychological logicians; for them everything is idea. For this reason, the proper conception of those judgements which

we express in English by "there is"[o] also eludes them. This existence is mixed up by Mr. B. Erdmann (*Logik* I, p. 311) with actuality, which, as we saw, is also not clearly distinguished from objectivity. Of what are we in fact asserting that it is actual when we say, there are square roots of Four? Is it Two or −2? But neither the one nor the other is in any way named. And if I wanted to say that the number Two acted or was active or actual, then this would be false and quite different from what I want to say with the proposition "There are square roots of Four". The confusion here before us is almost as bad as can be; since it does not involve concepts of the same level, but rather collapses a concept of the first level with a concept of the second. This is a hallmark of the obtuseness of psychological logic. Someone who has, generally, attained a more open point of view may wonder how such a mistake could be made by a professional logician; but before one can gauge the scale of such an error, one obviously has to recognise the distinction between concepts of first and second level in the first place, and psychological logic will presumably be incapable of that. The greatest barrier to this will be that the proponents are so exceedingly in awe of the psychological profundity, which however is nothing but psychological corruption of logic. And thus our thick logic books come about, bloated with unhealthy psychological lard, concealing all finer details. A fruitful cooperation between mathematicians and logicians is thereby rendered impossible. While the mathematician defines objects, concepts and relations, the psychological logician is listening in on the coming and going of ideas, and in the end the mathematician's defining can only appear foolish to him, since it does not convey the nature of ideas. He looks into his psychological peep box[p] and says to the mathematician: I see nothing at all of what you are defining. And the latter can merely answer: no wonder! For it is not where you are looking for it.

This may suffice to put my logical standpoint into a clearer light by the contrast. The distance from psychological logic seems to me to be as wide as the sky, so much so that there is no prospect that my book will have an effect on it immediately. My impression is that the tree that I have planted has to heave an incredible load of stone to make space and light for itself. Still, I will not give up all hope that my book will eventually aid the overthrow of psychological logic. To make the proponents of the latter come to terms with my book, some acknowledgement from the mathematicians will not come amiss. And indeed, I believe that I can expect some support from this

quarter, since the mathematicians have in the end to make common cause against the psychological logicians. As soon as the latter deign to engage with my book seriously, even if only in order to refute it, I shall take myself to have won. For the whole of part II is really a test of my logical convictions. It is from the outset unlikely that such a construction could be built on an insecure, defective basis. But if anyone has different convictions, let him try to build a similar construction on them and he will find, I believe, that it does not work, or at least that it does not work so well. And I could only acknowledge it as a refutation if someone indeed showed that a better, more enduring building can be erected on different basic convictions, or if someone proved to me that my basic principles lead to manifestly false conclusions. But no one will succeed in doing so. And so may this book, even if belatedly, contribute to a renaissance of logic.

Jena in July, 1893.

G. Frege

Table of contents